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2008 J. Phys. A: Math. Theor. 41 465402

(http://iopscience.iop.org/1751-8121/41/46/465402)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 465402 (8pp) doi:10.1088/1751-8113/41/46/465402

Real scalar fields on manifolds

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Received 23 June 2008
Published 16 October 2008
Online at stacks.iop.org/JPhysA/41/465402

Abstract
A generic theory of a single real scalar field is considered, and a simple method is presented for obtaining a class of solutions to the equation of motion. These solutions are obtained from a simpler equation of motion that is generated by replacing a set of the original coordinates by a set of generalized coordinates, which are harmonic functions in the spacetime. These ansatz solutions solve the original equation of motion on manifolds that are defined by simple constraints. These manifolds, and their dynamics, are independent of the form of the scalar potential. Some scalar field solutions, and manifolds upon which they exist, are presented for Klein–Gordon and quartic potentials as examples. Solutions existing on leaves of a foliated space may allow inferences of the characteristics expected of exact bulk solutions.

PACS numbers: 11.27.+d, 11.10.Lm

1. Introduction

Scalar fields play a prominent role in modern physical theories. Scalar potentials with multiple vacuum states allow for the possible existence of various types of topological and nontopological solitons, including kinks and domain walls, cosmic strings and magnetic poles [1, 2]. Scalar field interactions can give rise to networks of defects [3, 4] and nested defects [5], where one defect may form inside another (host) defect. Solitonic structures associated with scalar moduli are found in dilatonic and low energy string theories [6]. The many interesting types of scalar field phenomena serve to motivate the study of various kinds of scalar field theories and their solutions. Often, attention is focused on a simplified scenario where scalar fields depend upon only one or two coordinates, and solutions are easier to obtain and analyze [7]. Solutions to the equations of motion that depend on several variables are generally less accessible, but may contain a relatively rich structure.

Here, we present a simple ansatz allowing one to map a solution of fewer coordinate variables to one of the more coordinate variables. These ansatz solutions, however, are subject to a caveat, in that they solve the equation of motion only on a well-defined manifold, or set of manifolds, in the spacetime. The manifold(s) may consist of the entire spacetime, or may be in the form of hypersurfaces within the spacetime. For a space that is foliated by a set of surfaces, it seems natural to expect that the set of solutions on the various leaves of the foliation will give an indication of the mathematical and physical natures of an exact solution solving the equation of motion in the spacetime bulk. This may provide a way to extract information about complicated solutions of a scalar field theory that would be otherwise hard to obtain.

We consider a theory of a single real scalar field described by an action

$$S = \int \mathrm{d}^N x \sqrt{g} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \tag{1.1}$$

in an $N = D + 1$-dimensional spacetime with D spatial dimensions and $\mu = 0, \ldots, D$. A mostly negative metric is used with $g_{\mu\nu} = (+, -, -, \ldots, -)$ and $g = |\det g_{\mu\nu}|$. The metric $g_{\mu\nu}(x)$ is taken to be a nondynamical background field, and, for simplicity, we take fields and coordinates to be dimensionless. The equation of motion (EoM) is

$$\Box \phi = \nabla_\mu \partial^\mu \phi = -\frac{\partial V}{\partial \phi} = -V'(\phi). \tag{1.2}$$

This second-order DE can be difficult to solve, especially if there is a complicated potential $V(\phi)$ or a solution is sought where ϕ depends on more than one or two coordinate variables. We therefore consider a simplifying ansatz that will generate solutions to the EoM, but the solutions generated by the ansatz generally exist on some set of manifolds or hypersurfaces. For some cases, the manifold is the full spacetime. In other cases, a continuous set of hypersurfaces can foliate the spacetime, or a dynamical set of surfaces may move through the space. These manifolds can therefore span the spacetime in one way or another and thereby give some indication of at least, qualitative features that exact 'bulk' solutions (which may be hard to obtain directly) of the EoM may be expected to exhibit. These ansatz solutions form a subset of the full solution spectrum for the theory.

The ansatz is based on the idea that the function $\phi(x^\mu)$ can depend on the coordinates x^μ through a set of linearly independent functions $q^\alpha(x^\mu)$, where the number of functions q^α is less than or equal to the number of spacetime coordinates x^μ. The q^α serve as generalized coordinates, and must satisfy certain constraint conditions in order for $\phi[q^\alpha(x^\mu)]$ to satisfy the original EoM. These constraints, in turn, define some manifold of dimension $\leqslant N$ on which the solutions exist. These constraints are associated with a q-space metric, which has components that become Minkowski valued on the solution manifold. In addition, the functions q^α must be harmonic in the original spacetime, satisfying $\Box q^\alpha(x^\mu) = 0$. For the case where the q^α consists of just one spacelike generalized coordinate, say $q^1 = \xi(x^\mu)$, with $\phi = \phi[\xi(x^\mu)]$, the ansatz considered here reduces to a BPS-like ansatz where the solution $\phi(\xi)$ can be obtained directly from the potential function $V(\phi)$. The solution manifolds and their associated dynamics are independent of the form of the scalar field potential.

In the following sections we present the solution generating ansatz. Some concrete examples of solutions of scalar field theories, and manifolds on which they exist, are then presented. These include theories with potentials for massless and massive Klein–Gordon fields, as well as ϕ^4 theory. We focus on 1D and 2D cases, where ϕ depends upon only one or two q functions, respectively. For the 1D case the generalized coordinate can be either a timelike or a spacelike one. For the 2D case there can be one timelike and one spacelike function, or two that are spacelike. Static and dynamical solutions are obtained describing configurations such as Klein–Gordon fields, kinks and domain ribbons on various manifolds.

2. The ansatz

The purpose of our simplifying ansatz is to obtain solutions to the EoM in (1.2) by considering $\phi(x^\mu)$ to have a dependence on coordinates x^μ only through a set of linearly independent generalized coordinate functions $q^\alpha(x^\mu)$, i.e., $\phi(x^\mu) = \phi[q^\alpha(x^\mu)]$. The number of generalized coordinates q^α is less than, or equal to, the number of spacetime coordinates x^μ. In other words, the α indices can take any set of the values of the μ indices, where $\mu = 0, 1, 2, \ldots, D$. We could choose $q^\mu = x^\mu$ for some of the coordinates, but we will focus on the case where the number of $q^\alpha \neq x^\alpha$ is less than the total number of spacetime coordinates $\{x^\mu\}$, and therefore $\phi(q^\alpha)$ is a function of $M < N$ generalized coordinates $q^\alpha(x) \neq x^\alpha$.

Using a notation where differentiation with respect to a q coordinate is denoted by an overbar, $\bar{\partial}_\alpha = \partial/\partial q^\alpha$, we write

$$\partial_\mu \phi = (\partial_\mu q^\alpha)\bar{\partial}_\alpha \phi, \qquad \partial^\mu \phi = (\partial^\mu q^\alpha)\bar{\partial}_\alpha \phi, \qquad \bar{\partial}_\alpha \equiv \frac{\partial}{\partial q^\alpha}. \tag{2.1}$$

The term $\Box\phi$ on the left-hand side of (1.2) can be written as

$$\begin{aligned}\Box\phi &= \nabla_\mu \partial^\mu \phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial^\mu \phi) = \frac{1}{\sqrt{g}}\partial_\mu[\sqrt{g}(\partial^\mu q^\alpha)\bar{\partial}_\alpha \phi] \\ &= (\Box q^\alpha)\bar{\partial}_\alpha \phi + (\partial_\mu q^\alpha \partial^\mu q^\beta)\bar{\partial}_\alpha \bar{\partial}_\beta \phi.\end{aligned} \tag{2.2}$$

The EoM of (1.2) then takes the form

$$\Box\phi + V'(\phi) = (\Box q^\alpha)\bar{\partial}_\alpha \phi + (\partial_\mu q^\alpha \partial^\mu q^\beta)\bar{\partial}_\alpha \bar{\partial}_\beta \phi + V'(\phi) = 0. \tag{2.3}$$

We consider a class of solutions that satisfies the simplified EoM in the q-space,

$$\eta^{\alpha\beta}\bar{\partial}_\alpha \bar{\partial}_\beta \phi + V'(\phi) = 0 \tag{2.4}$$

subject to the conditions

$$\Box q^\alpha = 0, \qquad \partial_\mu q^\alpha \partial^\mu q^\beta = \eta^{\alpha\beta}. \tag{2.5}$$

The first condition requires $q^\alpha(x^\mu)$ to be a harmonic function, $\Box q^\alpha = \nabla_\mu \partial^\mu q^\alpha = 0$, and the second condition imposes a set of constraints upon the q^α. This set of constraints must be satisfied simultaneously. Each constraint equation can lead to a constraint between the coordinates x^μ, and can therefore define a manifold. The solution manifold $\mathcal{M}$ is the intersection of all of the individual constraint manifolds.

To summarize, we can generate a solution $\phi(x^\mu)$ of the EoM by considering a solution $\varphi(x^\alpha)$ that solves an equation of motion of the form $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\varphi + V'(\varphi) = 0$ in a Minkowski spacetime or Euclidean space, with φ depending on a set of coordinates x^α that is a subset of the spacetime coordinates x^μ. We then make replacements $x^\alpha \to q^\alpha(x^\mu)$ and $\varphi(x^\alpha) \to \phi[q^\alpha(x^\mu)]$ to obtain the q-space equation of motion in (2.4). This function $\phi(q^\alpha)$ will also be a solution to the original EoM in (1.2) on the manifold $\mathcal{M}$, provided that the conditions in (2.5) are satisfied. Each function $q^\alpha(x^\mu)$ is harmonic in the original spacetime, and the constraint equations $\partial_\mu q^\alpha \partial^\mu q^\beta = \eta^{\alpha\beta}$ define the solution manifold $\mathcal{M}$ where all constraints are satisfied simultaneously. Then the EoM is satisfied on $\mathcal{M}$, i.e.,

$$\{\nabla_\mu \partial^\mu \phi + V'(\phi)\}|_{\mathcal{M}} = 0. \tag{2.6}$$

Let us try to look at this in a slightly different way. Suppose that we have a spacetime with N coordinates x^μ and metric $g_{\mu\nu}(x)$. We then define N new generalized coordinates $q^\mu(x)$, although some of the q's may be identically equal to some of the x's; e.g., $q^m = x^m$, where $\{q^m\}$ is a proper subset of $\{q^\mu\}$. We then have nontrivial functions $q^\alpha(x)$ for a subset $\{q^\alpha\}$ $(\alpha \neq m)$. Now consider a diffeomorphism that takes $x^\mu \to q^\mu$ and the metric $g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(q)$.

A tensor transformation of the (contravariant) metric is $\bar{g}^{\rho\sigma}(q) = \partial_\mu q^\rho \partial_\nu q^\sigma g^{\mu\nu}(x)$. The constraint equations $\partial_\mu q^\alpha \partial^\mu q^\beta = \eta^{\alpha\beta}$ state that the $\alpha\beta$ components of $\bar{g}_{\rho\sigma}(q)$—a subset of the full set of $\{\bar{g}_{\rho\sigma}\}$—become Minkowski valued on the solution manifold $\mathcal{M}$. The solution $\phi(x^\mu)$ to the EoM is mapped into a function $\phi(q^\alpha)$, which solves a DE (on $\mathcal{M}$) with fewer (generalized) coordinate variables on a manifold $\mathcal{M}$ where some of the metric components $\bar{g}_{\rho\sigma}$ take Minkowski values.

3. Some illustrations

A few concrete illustrations are given for implementing the method described above. We focus on cases where there are only one or two q functions, i.e., the q^α-space (the number of q's on which ϕ depends) is one or two dimensional.

3.1. The 1D case

Spacelike case: let us seek a solution to the EoM involving one spacelike function, say $q^1 = \xi(x^\mu)$ so that the solution to the EoM $\Box\phi(x^\mu) + V'(\phi) = 0$ on the manifold $\mathcal{M}$ is given by $\phi[\xi(x^\mu)]$. The function ξ must be harmonic, $\nabla_\mu \partial^\mu \xi = \Box\xi = 0$, and must satisfy the constraint in (2.5) which takes the form

$$\partial_\mu \xi \partial^\mu \xi = -1. \tag{3.1}$$

Nonlinear harmonic functions ξ will solve this constraint when the coordinates x^μ are constrained, and thereby define a manifold $\mathcal{M}$. For example, consider the spacetime to be a 4D Minkowski spacetime, $g_{\mu\nu}(x) = \eta_{\mu\nu}$, and choose the harmonic function $\xi_R = xy/R$, where R is an arbitrary real, positive constant. Constraint (3.1) then becomes the condition

$$x^2 + y^2 = R^2, \tag{3.2}$$

so that the spatial surface $\mathcal{M}_R$ is a static cylinder of radius R centered on the z-axis. Then the solution to the EoM on $\mathcal{M}_R$, where $\xi_R = xy/R = R\sin\theta\cos\theta$ (with θ being the ordinary azimuth angle) is $\phi_R(\theta) = \phi_R(R\sin\theta\cos\theta)$. Since R is a continuous real parameter, there is a continuum of surfaces $\mathcal{M}_R$ (concentric cylinders) on which solutions ϕ_R to the EoM exist. The space is then foliated by a set of concentric cylindrical leaves, with a solution $\phi_R(\theta)$ defined on each leaf labeled by the parameter R. Looking at the leaf solutions as R ranges from zero to infinity can give a glimpse of qualitative features expected of an exact solution $\Phi(x^\mu)$ to the EoM $\Box\Phi + V'(\Phi) = 0$ that exists in the bulk of the spacetime, i.e., a solution that satisfies the EoM throughout the entire spacetime. (Each of these leaf solutions ϕ_R generally has a nonvanishing normal derivative $\hat{n}\cdot\nabla\phi$ on the surface $\mathcal{M}_R$ in addition to tangential derivatives along the surface. The solution $\phi(\xi_R)$ takes a value of $\phi_R(\xi_R) = \phi(\xi_R)|_{\mathcal{M}_R}$ on the surface $\mathcal{M}_R$ where ξ_R takes a value $\xi_R = (r^2/R)\sin\theta\cos\theta|_{r=R} = R\sin\theta\cos\theta$.)

This 1D case is an illustration of a 'BPS-like' ansatz, since the simplified equation in (2.4) is just

$$-\partial_\xi^2\phi(\xi) + V'(\phi) = 0 \tag{3.3}$$

and can be integrated to give

$$\tfrac{1}{\sqrt{2}}\partial_\xi\phi = \pm\sqrt{V + c}, \tag{3.4}$$

where c is an integration constant, determined by boundary conditions. The solution is then given by

$$\int \frac{\mathrm{d}\phi}{\sqrt{V + c}} = \pm\sqrt{2}(\xi - \xi_0) \tag{3.5}$$

which can be determined explicitly, once the form of the potential $V(\phi)$ is specified. The second-order EoM has been transformed into the first-order DE in (3.4), which resembles the DE for a BPS solution for a static field which is a function of the coordinate ξ. This BPS-like ansatz can be used to obtain new solutions on various manifolds for different scalar field theories. Specific examples follow. (We assume a 4D Minkowski spacetime.)

(1) *Lorentz boosted kink*: for a specific example, consider ϕ^4 theory with potential $V = (\phi^2 - 1)^2$. Choosing $c = 0$, (3.5) gives the familiar kink solution $\phi(\xi) = \tanh(\sqrt{2}\xi)$. Let us now choose a linear harmonic function, $\xi = a_\mu x^\mu$. Constraint (3.1) leads to $a_\mu a^\mu = -1$, which does not involve coordinates, but only constrains the constants a_μ. Therefore the solution manifold $\mathcal{M}$ is the full spacetime. Note that this choice of ξ includes a description of a Lorentz boost, as can be seen by choosing $a_0 = -\gamma u$, $a_1 = \gamma$, $a_2 = a_3 = 0$. The constraint has as a solution $\gamma = (1 - u^2)^{-1/2}$, which is the relativistic γ factor associated with a boost along the x-axis with velocity u. Then $\xi = \gamma(x - ut)$ gives a Lorentz transform from x to $x' = \xi(x, t)$. The kink solution $\phi(\xi)$ therefore can be written as $\phi(x, t) = \tanh[\sqrt{2}\gamma(x - ut)]$, a Lorentz boosted kink defined in the whole spacetime. (Linear functions q^α in a Minkowski spacetime generate constraints involving only constants, rather than coordinates. Nonlinear functions q^α are associated with coordinate-constrained manifolds.)

(2) ϕ^4 *domain ribbons on static cylinder*: as another example, consider ϕ^4 kink solutions on the surface of the cylinder of radius R in (3.2), generated by the function $\xi_R = xy/R = (r^2/R)\sin\theta\cos\theta$. On the surface $\mathcal{M}_R$ this takes the value $\xi_R|_{\mathcal{M}} = R\sin\theta\cos\theta$. The kink solutions $\phi(\xi) = \pm\tanh(\sqrt{2}\xi)$ on the cylinder surface $\mathcal{M}_R$ are

$$\phi_R(\xi_R) = \pm\tanh(\sqrt{2}R\sin\theta\cos\theta). \tag{3.6}$$

These are z independent solutions with zeros located on the $\pm x$- and $\pm y$-axes. The energy density is

$$T_{00} = g_{00}[2V] = \frac{2}{\cosh^4(\sqrt{2}R\sin\theta\cos\theta)}. \tag{3.7}$$

This energy density is maximized at the zeros of the solution ϕ; we can think of these solutions as domain ribbons on the cylinder, parallel to the z-axis. For either the (+) or (−) solutions, we have zeros of ϕ with positive slopes separated by zeros of ϕ with negative slopes in between. This leads us to interpret the solution as a set of four ribbon-like structures consisting of two ribbons separated by antiribbons in between.

As the parameter R ranges from zero to infinity, we infer from the $\{\phi_R(\xi_R)\}$ the existence of a static bulk solution $\Phi(x, y)$ describing perpendicular domain walls centered on the x- and y-axes, where $\Phi = 0$, with Φ entering vacuum states $\Phi = \pm 1$ away from the axes at asymptotic distances from the origin. The set of surface solutions $\{\phi_R\}$ presumably resemble intersections of a bulk solution Φ with the leaves of the $\{\mathcal{M}_R\}$ surfaces.

Timelike case: if we instead consider a single timelike generalized coordinate $\tau(x^\mu)$, the EoM reduces to $\partial_\tau^2\phi(\tau) + V'(\phi) = 0$ with the harmonic function τ subject to the constraint $\partial_\mu\tau\partial^\mu\tau = \eta^{00} = 1$. The DE for $\phi(\tau)$ can be solved once the form of the potential (along with boundary conditions) is specified. The manifold $\mathcal{M}$ is generated by the choice of τ and the constraint that it must satisfy.

K–G field on dynamical 2-branes: as an example, in a 4D Minkowski spacetime, a potential $V = \frac{1}{2}m^2\phi^2$ admits a simple solution $\phi(\tau) = \cos m\tau$. Choosing, for example, a function $\tau = xt$ leads to a constraint $x^2 - t^2 = 1$, which defines two parallel planes perpendicular to the x-axis, located by

$$x^\pm(t) = \pm\sqrt{t^2 + 1}. \tag{3.8}$$

The planes approach one another for $t < 0$, stop and turn around at $t = 0$, then move away from each other for $t > 0$. The value of $\tau^{\pm}$ on $\mathcal{M}^{\pm}$ is $\tau^{\pm} = x^{\pm}t = \pm t\sqrt{t^2 - 1} = \pm x^{\pm}\sqrt{(x^{\pm})^2 - 1}$. The solution $\phi(x, t)$ of the EoM can then be written, for instance, as

$$\phi(x, t) = \cos m\tau = \cos(mxt). \tag{3.9}$$

This function satisfies the EoM $\left(\partial_t^2 - \partial_x^2\right)\phi(x, t) + m^2\phi(x, t) = 0$ *when the EoM is evaluated on the manifold* $\mathcal{M}$. The value of the solution $\phi(x^{\pm}t)$ on $\mathcal{M}^{\pm}$ is then given by

$$\phi_{\mathcal{M}^{\pm}}(t) = \cos m\tau^{\pm} = \cos[mt\sqrt{t^2 - 1}]. \tag{3.10}$$

Keep in mind that it is not (3.10) that solves the EoM on $\mathcal{M}$, but rather the function in (3.9), which has nonvanishing normal derivatives (x derivatives). The solution of (3.9) then takes the value given by (3.10) on the surfaces $\mathcal{M}^{\pm}$ where $x = x^{\pm}$.

3.2. The 2D case

1+1 case: consider ϕ to be a function of just two q's, say a timelike function $q^0 = \tau(x^\mu)$ and a spacelike function $q^1 = \xi(x^\mu)$, so that $\phi = \phi(\tau, \xi)$. Then the conditions in (2.5) are given explicitly by the harmonic conditions $\Box\tau = \Box\xi = 0$ supplemented by the set of constraints

$$\begin{array}{lcl} \partial_\mu q^0\partial^\mu q^0 = \eta^{00} & & \partial_\mu\tau\partial^\mu\tau = 1 \\ \partial_\mu q^0\partial^\mu q^1 = \eta^{01} & \text{or} & \partial_\mu\tau\partial^\mu\xi = 0 \\ \partial_\mu q^1\partial^\mu q^1 = \eta^{11} & & \partial_\mu\xi\partial^\mu\xi = -1. \end{array} \tag{3.11}$$

This set of simultaneous constraints can, in general, lead to intersecting surfaces, etc, and the solution manifold, $\mathcal{M}$, is the common intersection of all the individual constraint manifolds. The scalar field $\phi[\tau(x^\mu), \xi(x^\mu)]$ is a solution of the simplified EoM

$$\left(\partial_\tau^2 - \partial_\xi^2\right)\phi + V'(\phi) = 0 \tag{3.12}$$

and this solution solves the original EoM $\nabla_\mu\partial^\mu\phi + V'(\phi) = 0$ on the solution manifold $\mathcal{M}$. We give specific examples below. (We assume a flat 4D spacetime.)

(1) *Massless scalar field*: for a potential $V(\phi) = 0$ the general solution of (3.12) is

$$\phi(\tau, \xi) = F(\tau + \xi) + G(\tau - \xi), \tag{3.13}$$

where F and G are arbitrary functions of the indicated arguments and $\tau(x^\mu)$ and $\xi(x^\mu)$ are functions that satisfy (3.11). An example of such τ and ξ functions is

$$\tau = \sqrt{2}t - z, \qquad \xi = xy = r^2\sin\theta\cos\theta \tag{3.14}$$

for which $\mathcal{M}$ is a static cylinder of unit radius centered on the z-axis. Then on the cylindrical surface $\mathcal{M}$ the solution in (3.13) takes the form

$$\phi(\tau, \xi)|_{\mathcal{M}} = \phi_{\mathcal{M}}(t, z, \theta) = F(\sqrt{2}t - z + \sin\theta\cos\theta) + G(\sqrt{2}t - z - \sin\theta\cos\theta). \tag{3.15}$$

These running waves have the form $f(\sqrt{2}t - \zeta_\pm)$, with $\zeta_\pm = z \pm \sin\theta\cos\theta$.

(2) *Massive Klein–Gordon field*: for a potential $V(\phi) = \frac{1}{2}m^2\phi^2$ a simple wavelike solution of (3.12) is

$$\phi = \cos(\omega\tau - k\xi), \qquad \omega^2 = k^2 + m^2. \tag{3.16}$$

We choose the same manifold functions as before, given in (3.14). The ansatz solution is then

$$\phi = \cos[\omega(\sqrt{2}t - z) - kr^2\sin\theta\cos\theta] \tag{3.17}$$

and on the cylinder $\mathcal{M}$ we set $r = 1$. We could write this as $\phi_{\mathcal{M}} = \cos[\Omega t - Kz + \delta(\theta)]$, with $\Omega = \sqrt{2}\omega$, $K = \omega$, and phase parameter $\delta(\theta) = -k\sin\theta\cos\theta$. The condition $\omega^2 - k^2 = m^2$ gives

$$\Omega^2 - K^2 = \omega^2 = k^2 + m^2 \equiv M^2. \tag{3.18}$$

So (3.17) and (3.18) describe a massive plane wave traveling in the z-direction on the cylinder, with energy Ω, momentum K and effective mass $M = \sqrt{k^2 + m^2}$. There is an angular-dependent phase constant $xy = \sin\theta\cos\theta$ which vanishes on the x- and y-axes, but becomes nonzero elsewhere.

(3) *Dynamical ϕ^4 domain ribbons*: for a potential $V(\phi) = (\phi^2 - 1)^2$ we displayed a static solution for a kink as $\phi(\xi) = \tanh(\sqrt{2}\xi)$ for the 1D case above. For a simple 2D solution satisfying (3.12) we take a Lorentz boosted version of $\phi(\xi)$, with $\xi \to \gamma(\xi - u\tau)$, which we write as

$$\phi(\tau, \xi) = \tanh[\sqrt{2}\gamma(\xi - u\tau)]. \tag{3.19}$$

We again choose the functions τ and ξ in (3.14). The ansatz solution on the cylinder then takes the form

$$\phi(\tau, \xi)|_{\mathcal{M}} = \tanh\{\sqrt{2}\gamma[\sin\theta\cos\theta - u(\sqrt{2}t - z)]\}. \tag{3.20}$$

For $u = 0$, $\gamma = 1$ this describes a pair of domain ribbons, each ribbon separated from the next by an antiribbon, all lying parallel to the z-axis and centered on the $\pm x$- and $\pm y$-axes, where the energy density maximizes (at $\phi = 0$, or $xy = 0$). However, for $u \neq 0$ the zeros of ϕ are shifted to positions located by $xy = \sin\theta\cos\theta = u(\sqrt{2}t - z)$, indicating that the locations of the ribbon cores on the cylinder wall become z and t dependent dynamical objects. For instance, at the time $t = 0$ we have ribbons localized at $xy = \sin\theta\cos\theta = -uz$ so that the ribbons appear to wind around the cylinder in a helical fashion, and these windings move as t progresses.

2+0 case: now consider a type of solution where ϕ depends on two spacelike generalized coordinates $q^1 = \xi(x^\mu)$ and $q^2 = \sigma(x^\mu)$. The equation of motion in (2.4) becomes

$$\left(\partial_\xi^2 + \partial_\sigma^2\right)\phi(\xi, \sigma) = V'(\phi) \tag{3.21}$$

with $\Box\xi = \Box\sigma = 0$. The constraints in (2.5) take the form

$$\begin{array}{lcl} \partial_\mu q^1 \partial^\mu q^1 = \eta^{11} & & \partial_\mu \xi \partial^\mu \xi = -1 \\ \partial_\mu q^1 \partial^\mu q^2 = \eta^{12} & \text{or} & \partial_\mu \xi \partial^\mu \sigma = 0 \\ \partial_\mu q^2 \partial^\mu q^2 = \eta^{22} & & \partial_\mu \sigma \partial^\mu \sigma = -1. \end{array} \tag{3.22}$$

Laplace's equation on a cylinder: example constraint functions are

$$\xi = xy = r^2 \sin\theta\cos\theta, \qquad \sigma = \gamma(z - ut), \qquad \gamma = 1/\sqrt{1 - u^2} \tag{3.23}$$

which describe Lorentz boosts in the z-direction on the surface of a cylinder of unit radius, centered on the z-axis. As an example of a potential, we choose that of a massless scalar field, $V(\phi) = 0$. In this case a general solution to (3.21) can be written as

$$\phi(\xi, \sigma) = \sum_k A_k \, \mathrm{e}^{-k\xi} \cos k\sigma. \tag{3.24}$$

For the ξ and σ chosen above, the solution on the cylinder becomes

$$\phi_{\mathcal{M}} = \sum_k A_k \, \mathrm{e}^{-k\sin\theta\cos\theta} \cos k\gamma(z - ut). \tag{3.25}$$

Each k solution varies in a periodic way around the cylinder in the θ-direction, and is also a periodic function of $z - ut$. The values of k and the constants A_k are determined by boundary conditions.

4. Summary

A method has been presented which allows a class of nontrivial solutions to the equation of motion for a real scalar field $\phi(x^\mu)$, given by $\Box\phi + V'(\phi) = 0$, to be obtained from a simplified equation of motion. This is accomplished by replacing coordinate variables x^α on which a scalar field φ depends with generalized coordinates $q^\alpha(x^\mu)$, which are harmonic functions of coordinates x^μ. The function $\varphi(x^\alpha)$ satisfies the simpler equation $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\varphi(x) + V'(\varphi) = 0$, with the $\{x^\alpha\}$ being a subset of the full set of coordinates $\{x^\mu\}$. The replacements $x^\alpha \to q^\alpha$ and $\varphi(x^\alpha) \to \phi(q^\alpha)$ result in a function $\phi(x^\mu) = \phi[q^\alpha(x^\mu)]$ that solves the original EoM $\nabla_\mu\partial^\mu\phi(x) + V'(\phi) = 0$, provided that a set of simple constraints is satisfied. These constraints give rise to spacetime manifolds $\mathcal{M}$ on which the solution $\phi(x^\mu)$ exists. In a Minkowski spacetime, linear functions $q^\alpha(x^\mu)$ are associated with a manifold which is the full spacetime, with constraints on the constants, whereas for nonlinear functions $q^\alpha(x^\mu)$ the manifold is a subspace or hypersurface of the spacetime. Neither the manifolds nor their dynamics depend upon the form of the scalar field theory. Examples of manifolds and solutions for different scalar field theories have been provided for the 1D and 2D cases, i.e., where the function ϕ depends on only one or two generalized coordinate functions q^α. Dynamical manifolds, or a continuum of static manifolds, can span the bulk of the spacetime, allowing some inference of the nature of exact bulk solutions $\Phi(x^\mu)$ that solve the EoM throughout the entire spacetime, without being restricted to any particular manifold.

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